Chapter 1
Mean Variance Portfolio Theory

This book is about portfolio construction and risk analysis in the real-world context where optimization is done with constraints and penalties specified by the portfolio manager or fund mandate. Consequently all useful portfolio optimization requires the use of numerical optimization methods. In addition to treating constraints and penalties, modern methods of portfolio optimization included in this book move beyond the classic portfolio mean-variance optimization (MVO) theory introduced by Markowitz (1952, 1959) to deal with non-normality of returns using down-side risk measures. None-the-less the MVO theory is well-established, both in academic courses on portfolio optimization where it provides the foundation for the capital asset pricing model (CAPM), and as the basis of commercial portfolio optimization products. Furthermore MVO portfolios form ”ideal” reference points for constrained MVO portfolios, and constrained MVO portfolios form a reference point for constrained mean versus downside risk optimal portfolios. Thus this chapter is devoted to providing a brief treatment of the the MVO theory in several important contexts.

The basic MVO theory makes the highly idealized assumption that there is no constraint on short-selling and that only simple linear equality constraints on weights are imposed. The latter include for example the full-investment constraint that the portfolio weights sum to one, and a dollar-neutral constraint that the portfolio weights sum to zero. With these assumptions analytic formulas for optimal portfolio weights, mean return and variance are obtainable using the method of Lagrange multipliers. We cover two main application contexts: (1) Absolute returns portfolios, i.e., no benchmark relative concerns, and (2) Benchmark relative (active) portfolios. Case (1) contains two important sub-cases: (1a) The investment universe is taken to be a combination of cash and risky assets, where *cash* is taken to mean investment in a risk-free asset such as 90 day T-bills in the U.S., and (1b) The investment universe consists of only risky assets. Case (1a) contains dollar neutral portfolios and market neutral portfolios as special cases.

In each of the two main cases above, the optimization problems themselves are treated in two equivalent ways: (a) Minimization of risk subject to achieving a target portfolio mean return (or its infrequently treated dual consisting of maximizing mean return subject to a constraint on risk), and (b) Maximization of a quadratic utility function. The former is a traditional academic way of treating the problem, and the latter is the more common viewpoint taken in commercial portfolio optimization and risk management software products. We describe both approaches in this chapter.

1.1 Portfolio Mean and Variance

This section provides definitions and notation that will be used extensively throughout the book. The basic context is that you have a portfolio of n assets with returns r_{ti}, $i = 1, 2, \cdots, n$, $t = 1, 2, \cdots, T$, and portfolio weights w_i, $i = 1, 2, \cdots, n$. The latter are assumed to be constant for the time interval $t = 1, 2, \cdots, T$. The portfolio weight w_i represents the fraction of total wealth V invested in asset i, i.e., the wealth invested in asset i is Vw_i. Unless otherwise noted the returns are assumed to be arithmetic, i.e., $r_{ti} = (p_{t,i} - p_{t-1,i})/p_{t-1,i} = p_{t,i}/p_{t-1,i} - 1$. In vector notation the asset returns are

$$\mathbf{r}_t = (r_{t1}, ..., r_{tn})', t = 1, ..., T \tag{1.1}$$

and the portfolio weights are

$$\mathbf{w} = (w_1, w_2, \cdots, w_n)'. \tag{1.2}$$

The portfolio return at time t is

$$r_{P,t} = \sum_{i=1}^{n} w_i r_{ti} = \mathbf{w}'\mathbf{r}_t \tag{1.3}$$

and we often drop the subscript t and write $r_P = \mathbf{w}'\mathbf{r}$.

The key quantities in portfolio MVO are the portfolio mean return

$$\mu_P = E(r_P) \tag{1.4}$$

and the portfolio variance

$$\sigma_P^2 = \text{var}(r_P) = E(r_P - \mu_P)^2. \tag{1.5}$$

The key expressions we need are those for the portfolio mean and variance in terms of the asset mean returns and covariances. The notation for asset mean returns is

$$\begin{aligned}\boldsymbol{\mu} &= E(\mathbf{r}_t)\\ &= (E(r_{t1}),E(r_{t2}),\cdots,E(r_{tn}))'\\ &= (\mu_1,\mu_2,\cdots,\mu_n)'\end{aligned} \tag{1.6}$$

where we have assumed that the means do not vary with time. The pairwise covariances of the asset returns are

$$\begin{aligned}\Sigma_{ij} &= E\left[(r_{ti}-\mu_i)(r_{tj}-\mu_j)\right], i,j=1,\cdots,n\\ &= E(r_{ti}r_{tj})-\mu_i\mu_j\end{aligned} \tag{1.7}$$

where we have also assumed that the covariances do not change over time, and the $n\times n$covariance matrix of the asset returns is

$$\begin{aligned}\boldsymbol{\Sigma} &= E\left[(\mathbf{r}_t-\boldsymbol{\mu})(\mathbf{r}_t-\boldsymbol{\mu})'\right]\\ &= E(\mathbf{r}_t\mathbf{r}'_t)-\boldsymbol{\mu}\boldsymbol{\mu}'.\end{aligned} \tag{1.8}$$

The pairwise correlations are given by

$$\begin{aligned}\rho_{ij} &= \frac{\text{cov}(r_i,r_j)}{\text{var}^{1/2}(r_i)\cdot\text{var}^{1/2}(r_j)}, i,j=1,\cdots,n\\ &= \frac{\text{cov}(r_i,r_j)}{\sigma_i\cdot\sigma_j}\\ &= \frac{\Sigma_{ij}}{\Sigma_{ii}^{1/2}\cdot\Sigma_{jj}^{1/2}}\end{aligned} \tag{1.9}$$

and the $n\times n$ correlation matrix is

$$\mathbf{R} = diag(\sigma_i^{-1})\cdot\Sigma\cdot diag(\sigma_i^{-1}). \tag{1.10}$$

With the above notation the portfolio mean return is

$$\begin{aligned}\mu_P &= E\left(r_P\right)\\ &= E(\mathbf{w}'\mathbf{r})\\ &= \mathbf{w}'E(\mathbf{r})\\ &= \mathbf{w}'\mu\end{aligned} \tag{1.11}$$

and the portfolio variance is

$$\begin{aligned}\sigma_P^2 &= \text{var}(r_P)\\ &= \text{var}(\mathbf{w}'\mathbf{r})\\ &= E(\mathbf{w}'\left(\mathbf{r}-\mu\right))^2\\ &= E(\mathbf{w}'\left(\mathbf{r}-\mu\right)\left(\mathbf{r}-\mu\right)'\mathbf{w})\\ &= \mathbf{w}'\Sigma\mathbf{w}\end{aligned} \tag{1.12}$$

When we use the term *volatility* of an asset we are referring to the standard deviation of the asset's returns. So the volatility of a portfolio is

$$\sigma_P = \left(\mathbf{w}'\Sigma\mathbf{w}\right)^{1/2}. \tag{1.13}$$

We note that the portfolio variance expression is a quadratic form in the portfolio weights. Unless noted otherwise we assume that the covariance matrix of asset returns is positive definite, which means that the portfolio variance and volatility are positive for any non-zero weight vector $\mathbf{w}$. It also means that Σ is non-singular and has an inverse Σ^{-1} which is positive definite and non-singular. For details see APPENDIX A. We shall encounter the use of Σ^{-1} frequently.

1.2 Global Minimum Variance Portfolios

The global minimum variance (GMV) portfolio is a special case of minimum variance portfolios that contain only risky assets and satisfy the *full-investment* constraint that the portfolio weights sum to one, but there is no other constraint and in particular no limit on short sales. We begin by deriving the analytic formula for a GMV portfolio for two reasons. One the one hand the derivation illustrates the simplest use of the method of Lagrange multipliers to obtain an analytic solution. On the other hand GMV portfolios, possibly with diversification inducing weights constraints, are of interest as index alternatives to market capitalization weighted indexes.

Let $\mathbf{1} = (1, 1, \cdots, 1)'$ be the unit vector of n ones. The GMV portfolio weight vector is the solution of:

$$\min_{\mathbf{w}} \sigma_P^2(\mathbf{w}) = \min_{\mathbf{w}} \mathbf{w}'\Sigma\mathbf{w}$$

subject to

$$\mathbf{w}'\mathbf{1} = 1.$$

The Lagrangian for this minimization problem is

$$L(\mathbf{w}) = \frac{1}{2}\mathbf{w}'\Sigma\mathbf{w} + \lambda\left(1 - \mathbf{w}'\mathbf{1}\right) \tag{1.14}$$

and setting the derivative of the Lagrangian equal to zero gives

$$\Sigma\mathbf{w} - \lambda\mathbf{1} = 0.$$

which gives the form of the optimal weight vector in terms of the unknown Lagrange multiplier:

$$\mathbf{w} = \lambda^{-1}\Sigma^{-1}\mathbf{1}.$$

Solving the constraint equation for the Lagrange multiplier gives $\lambda = (\mathbf{1}'\Sigma^{-1}\mathbf{1})^{-1}$ and so

$$\mathbf{w}_{GMV} = \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}'\Sigma^{-1}\mathbf{1}}. \tag{1.15}$$

Using the general expression (1.11) for portfolio mean return gives

$$\mu_{GMV} = \mathbf{w}'_{GMV}\mu = \frac{\mathbf{1}'\Sigma^{-1}\mu}{\mathbf{1}'\Sigma^{-1}\mathbf{1}} \tag{1.16}$$

and using the general expression (1.12) for portfolio variance gives

$$\sigma^2_{GMV} = \mathbf{w}'_{GMV}\Sigma\mathbf{w}_{GMV} = (\mathbf{1}'\Sigma^{-1}\mathbf{1})^{-1}. \tag{1.17}$$

Correspondingly the volatility of the GMV portfolio is

$$\sigma_{GMV} = (\mathbf{1}'\Sigma^{-1}\mathbf{1})^{-1/2}. \tag{1.18}$$

Note that the denominator of the expression (1.16) is a quadratic form based on the positive definite matrix Σ^{-1} and so it is always positive. However, this is not the case for the numerator which is a linear form and in general can be positive or negative. Thus the mean return of the global minimum variance portfolio will be positive if and only if $\mathbf{1}'\Sigma^{-1}\mu > 0$.

Recent years has seen interest in using GMV portfolios, calculated numerically with weights constraints as we discuss in the next chapter, as index alternatives to market capitalization indexes. For example Clarke et. al. (2006) show that such a portfolio can outperform a cap-weighted market index. Their empirical study is based on the following conditions:

- Monthly portfolio rebalancing 1968 through 2004 (456 months)
- Training window is one year of trailing daily excess returns
- 1,000 largest market cap stocks for each rebalance period
- Shrink 1,000 x 1,000 covariance matrix toward two-parameter covariance matrix (Ledoit and Wolf, 2004)
- Market is the cap-weighted portfolio of the 1,000 stocks
- GMV portfolio is long-only with upper bound of 3% on weights

Their results include the following where the mean returns are in excess of the T-bill rate which averaged 5.95%:

Portfolio	**Mean Return**	**Volatility**	**Sharpe Ratio**
Market	5.6%	15.4%	.36
GMV	6.5%	11.7%	.55

Other related references include:

- Scherer, B. (2011). A New Look at Minimum Variance Investing. Journal of Empirical Finance.
- Clarke, de Silva and Thorley, S. (2011). Minimum-Variance Portfolio Composition, Jour. of Portfolio Management, Fall, 37, No. 2, 10-24.
- Haugen, and Baker, N. (1991). The Efficient Market Inefficiency of Capitalization Weighted Stock Portfolios, Jour. of Portfolio Management, Spring, 35-40.

1.3 MVO Portfolios with Cash and Risky Assets

We now show how to obtain a minimum variance portfolio that contains both cash and risky assets subject to just two constraints, a mean-return constraint and the full-investment constraint. Once we have the optimal weights it is straightforward to obtain the mean return and variance of such a portfolio. What is perhaps surprising and important is that the relationship between the portfolio mean excess return and volatility is linear. There are two ways to solve this problem:

MinVar Minimize portfolio variance for specified mean return
QU Maximize quadratic utility for specified risk aversion

Both methods yield the same linear relationship between the optimal portfolio mean return and volatility. Since the QU approach is quite familiar to practitioners we derive the optimal weights that way, and leave the MinVar method to Problem xx.

Let w_0 represent the fraction of wealth invested in cash, i.e., invested in a risk-free asset with risk-free rate r_f, and let $\sum_{i=1}^{n} w_i = \mathbf{w}'\mathbf{1}$ represent the fraction of wealth invested in risky assets (stocks, bonds, ETF's, etc.). Note that the portfolio mean return is then $\mu_P(\mathbf{w}) = w_0 r_f + \mathbf{w}'\mu$. Thus the problem is to maximize the quadratic utility

$$\begin{aligned} Q(\mathbf{w}) &= \mu_P(\mathbf{w}) - \tfrac{1}{2}\lambda\sigma_P^2(\mathbf{w}) \\ &= w_0 r_f + \mathbf{w}'\mu - \tfrac{1}{2}\lambda\mathbf{w}'\Sigma\mathbf{w} \end{aligned} \tag{1.19}$$

where $\lambda > 0$ is a *risk aversion* parameter, subject to the constraint

$$w_0 + \mathbf{w}'\mathbf{1} = 1. \tag{1.20}$$

The above problem places no limit on short selling. Since the asset mean excess returns are $\mu_e = \mu - \mathbf{1}r_f$, we see that

$$\begin{aligned} \mu_P(\mathbf{w}) &= w_0 r_f + \mathbf{w}'\mu \\ &= w_0 r_f + \mathbf{w}'(\mu_e + \mathbf{1}r_f) \\ &= w_0 r_f + \mathbf{w}'\mathbf{1}r_f + \mathbf{w}'\mu_e \\ &= r_f + \mathbf{w}'\mu_e. \end{aligned} \tag{1.21}$$

Thus we need to maximize

$$r_f + \mathbf{w}'\mu_e - \frac{1}{2}\lambda\mathbf{w}'\Sigma\mathbf{w} \tag{1.22}$$

subject to the constraint (1.20), and since r_f is fixed it can be omitted from the objective. Problem **1.1** shows that the solution is

$$\mathbf{w}_{opt} = \lambda^{-1}\Sigma^{-1}\mu_e \tag{1.23a}$$
$$w_0 = 1 - \mathbf{w}'_{opt}\mathbf{1}. \tag{1.23b}$$

We see that as risk aversion $\lambda \to \infty$ the portfolio consists of an all cash position, and as risk aversion $\lambda \to 0$ the portfolio weights become unbounded in absolute value. Note that λ^{-1} is a *risk tolerance* parameter. As risk tolerance goes to zero the portfolio consists of an all cash position, and as risk-tolerance goes to infinity the portfolio weights become unbounded. In Problem **1.2** you explore what happens to the cash position in the latter case.

It is useful to express the optimal weights of the risky assets in terms of the portfolio mean excess returns $\mu_{P,e}$ as follows. From (1.21) it follows that

$$\begin{aligned} \mu_{P,e} &= \mu_P(\mathbf{w}) - r_f \\ &= \mathbf{w}'_{opt}\mu_e \\ &= \lambda^{-1}\mu'_e\Sigma^{-1}\mu_e \end{aligned} \tag{1.24}$$

is positive, and substituting the resulting expression for λ^{-1} into (1.23a) gives

$$\mathbf{w}_{opt} = \frac{\Sigma^{-1}\mu_e}{\mu_e\Sigma^{-1}\mu_e} \cdot \mu_{P,e}. \tag{1.25}$$

The variance of the optimal portfolio is

$$\begin{aligned} \sigma^2_{opt} &= \mathbf{w}'_{opt}\Sigma\mathbf{w}_{opt} \\ &= \frac{1}{\mu'_e\Sigma^{-1}\mu_e} \cdot \mu^2_{P,e} \end{aligned} \tag{1.26}$$

and since $\mu_{P,e} > 0$ the portfolio mean excess return and volatility are linearly related:

$$\mu_{P,e} = \sqrt{\mu'_e\Sigma^{-1}\mu_e} \cdot \sigma_{opt}. \tag{1.27}$$

We state the above result in slightly different form as theorem.

Theorem 1.1. *Quadratic utility optimal portfolios that contain cash and risky asset have the following mean return versus volatility relationship:*

$$\mu_P = r_f + \sqrt{\mu'_e\Sigma^{-1}\mu_e} \cdot \sigma_P. \tag{1.28}$$

Sharpe Ratio and Risk Aversion Parameter

The Sharpe ratio of any portfolio P is defined as the ratio of the portfolio mean excess return to the portfolio standard deviation:

$$SR_P \triangleq \frac{\mu_{P,e}}{\sigma_P}.$$

From (1.28) we see that the Sharpe ratio is constant along the efficient frontier with values given by

$$SR_{opt} = \frac{\mu_{opt,e}}{\sigma_{opt}} = \sqrt{\boldsymbol{\mu'}_e \Sigma^{-1} \mu_e} \tag{1.29}$$

The following code makes use of the functions `barplot.wts.R` and `mathEfrontCashRisky.R` to generate Figures 1.1, 1.2 and 1.3:

```
> library(xts)
> load("Computing/crsp.short.Rdata")
> source("Computing/mathEfrontCashRisky.R")
> source("Computing/barplot.wts.R")
> returns = midcap.ts[,1:10]

> plot.zoo(returns,plot.type = "multiple",main = "MID-CAP RETURNS")

> mathEfrontCashRisky(returns, scalex = 1.2, scaley = 2, bar.ylim = c(-1.5,3.5))

$MU.EQ.WT
[1] 0.01616507

$STDEV.EQ.WT
[1] 0.06004073

$SR.EQ.WT
[1] 0.2692351

$SR.EFRONT
[1] 0.3608565
```

The code computes the mean return, volatility and Sharpe ratio of an equally weighted portfolio, and the efficient frontier optimal Sharpe ratio given by (1.29). Figure 1.2 shows the position of the equally-weighted portfolio with the triangle symbol.

As for the risk aversion parameter note that equation (1.24) gives

$$\lambda = \frac{\mu'_e \Sigma^{-1} \mu_e}{\mu_{P,e}}$$

and (1.26) gives

$$\mu'_e \Sigma^{-1} \mu_e = \frac{\mu^2_{opt,e}}{\sigma^2_{opt}}.$$

Noting the expression for the optimal Sharpe ratio in (1.29) gives the following general expression for the risk aversion parameter along the efficient frontier:

$$\lambda = \frac{\mu_{opt,e}}{\sigma^2_{opt}} = \frac{1}{\sigma_{opt}} SR_{opt} \tag{1.30}$$

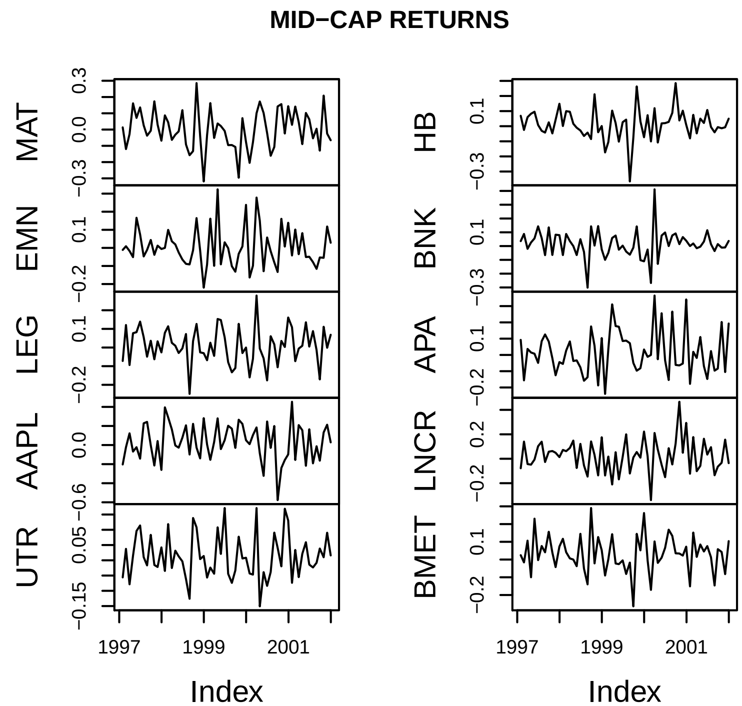

Fig. 1.1 Mid-Cap Stock Returns 1997-2002

You can plot the efficient frontier versus risk tolerance with the following code:

```
> library(xts)
> load("Computing/crsp.short.Rdata")
> source("Computing/mathEfrontCashRisky.R")
> source("Computing/barplot.wts.R")
> returns = midcap.ts[,1:10]
> mathEfrontCashRisky(returns, scalex = 1.2, scaley = 2, risk.tol = T, wts.plot = F)
```

There remains an interesting question. The efficient frontier was derived with the assumption that there is a mix of cash and risky assets. One might think that as a limiting case this efficient frontier should contain a position that is totally in risky assets. In order for that to be the case the condition $\mathbf{1}'\mathbf{w}_{opt} = 1$ must hold, which implies that

$$\mathbf{1}'\mathbf{w}_{opt} = \lambda^{-1}\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_e = 1.$$

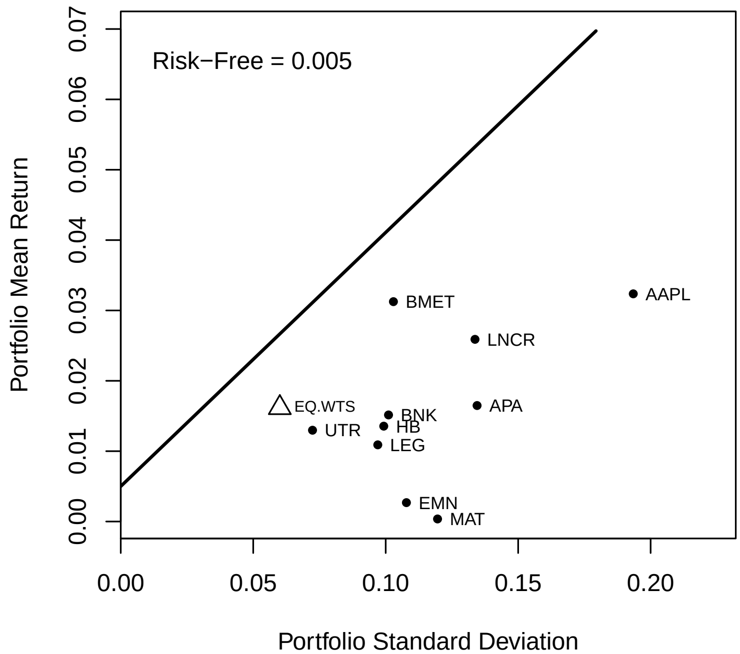

Fig. 1.2 Efficient Frontier with Cash and Risky Assets

Recalling that the risk aversion parameter λ is non-negative, we see that in order to have a full investment in risky assets position on the efficient frontier it must be that

$$\lambda = \mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_e > 0. \tag{1.31}$$

For the mid-cap returns efficient frontier of 1.4 the resulting risk aversion parameter λ turns out the be positive as the following computation shows:

```
> library(xts)
> load("Computing/crsp.short.Rdata")
> rf = .005
> returns = midcap.ts[,1:10]
> C = var(returns)
> mu.stocks = apply(returns, 2, mean)
> mue = mu.stocks - rf
> a = solve(C, mue)
> lambda = sum(a)
> lambda      # Risk aversion value
```

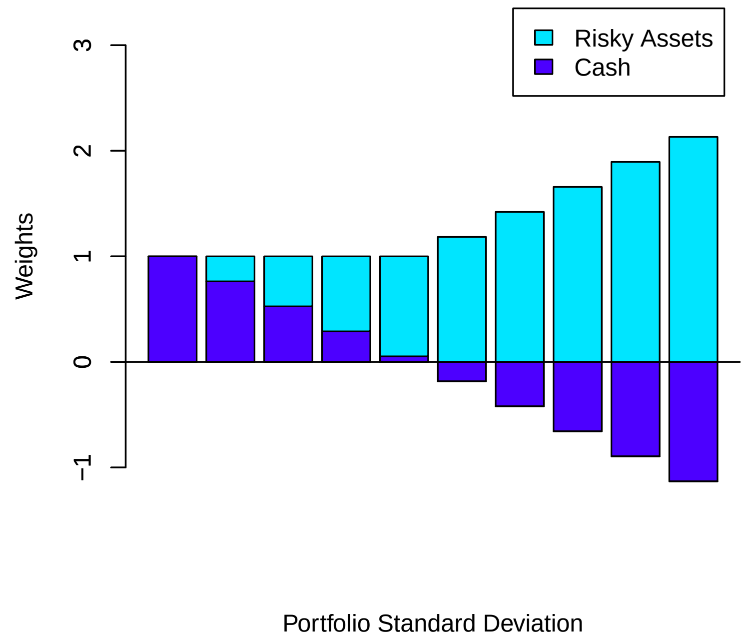

Fig. 1.3 Efficient Frontier with Cash and Risky Assets

```
[1] 4.287556
> 1/lambda    # Risk tolerance value
[1] 0.2332331
```

But the condition $\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_e > 0$ does not always hold. One wonders what simple condition, if any, would imply this condition. This question turns out to have a simple answer. Recalling the expression (1.16) for the mean return of the GMV portfolio, we see that the mean excess return of the GMV portfolio is:

$$\begin{aligned} \mu_{GMV,e} &= \frac{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}} - r_f \\ &= \frac{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}} - \frac{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}\cdot r_f}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}} \\ &= \frac{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_e}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}}. \end{aligned} \tag{1.32}$$

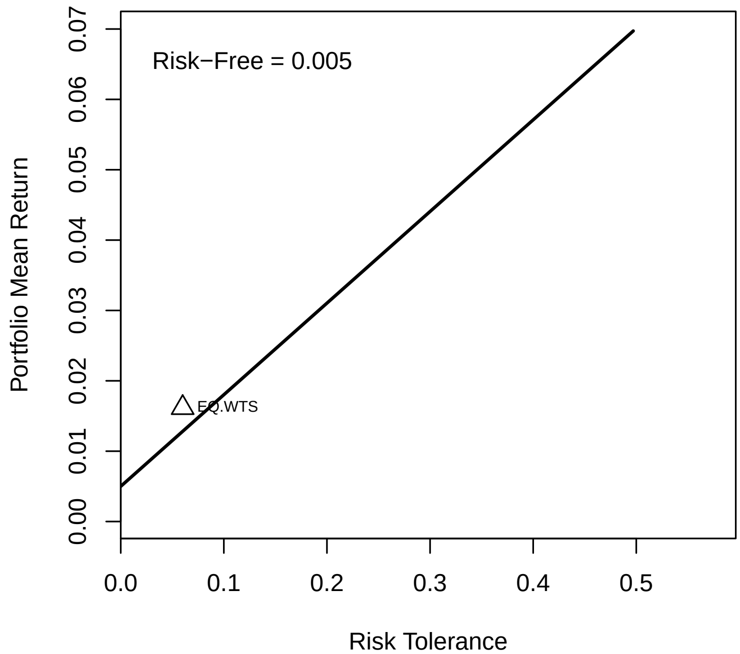

Fig. 1.4 Cash and Risky Assets Efficient Frontier versus Risk Tolerance

The condition for the mean excess return of the GMV portfolio to be positive is the same as the condition (1.31). In summary: A full investment in risky assets position exists on the efficient frontier for cash and risky assets if, and only if, the mean excess return of the GMV portfolio is positive. It is important to know that in practice the latter condition is not always satisfied. ** ADD HISTORICAL DATA EXAMPLE HERE, E.G., IN THE HIGH

INTEREST RATE PERIOD OF THE 70'S WE CAN LIKELY FIND SUCH CONDITIONS ARE FREQUENT **

** ADD GENERAL SIMPLE EXPRESSION FOR RISK TOLERANCE ALONG EFFICIENT FRONTIER **

1.4 MVO Portfolios with Risky Assets Only

1.5 Dollar Neutral and Market Neutral Portfolios

1.6 Benchmark Relative Optimization

1.7 Benchmark Relative Optimization

1.8 Liabilities and Surplus Efficient Frontier

1.9 Estimation Error and Portfolio Uncertainty

Problems

1.1. Use the method of Lagrange multipliers to show that maximization of (1.22) subject to the constraint (1.20) yields the expressions in (1.23a) and (1.23b).

1.2. Explain what happens to w_0 in (1.23b) as $\boldsymbol{\lambda} \to 0$ in (1.23).

1.3. Formulate the problem of finding the mean-variance optimal portfolio of cash plus risky assets as one of minimizing the portfolio variance for a specified portfolio mean excess return. Solve for the optimal weight vector using the method of Lagrange multipliers, and confirm that the result is identical to equation (1.23a) obtained by maximizing quadratic utility.